

Correlations of chaotic eigenfunctions: a semiclassical analysis

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Abstract

We derive a semiclassical expression for an energy smoothed autocorrelation function defined on a group of eigenstates of the Schrödinger equation. The system we considered is an energy-conserved Hamiltonian system possessing time-invariant symmetry. The energy smoothed autocorrelation function is expressed as a sum of three terms. The first one is analogous to Berry's conjecture, which is a Bessel function of the zeroth order. The second and the third terms are trace formulae made from special trajectories. The second term is found to be direction dependent in the case of spacing averaging, which agrees qualitatively with previous numerical observations in high-lying eigenstates of a chaotic billiard.

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I. INTRODUCTION

With the intention of finding a fingerprint of classical chaos in quantum systems, the study of eigenstates has attract increasing attentions both theoretically [1–20] and experimentally [21,22] over the last two decades. Among many others, the statistics of wavefunction amplitude $|\psi(\mathbf{r})|^2$ and two-point correlation function $\langle \psi(\mathbf{r}')\psi(\mathbf{r}'') \rangle$ have been widely studied. Recently, the study has been extended to the statistics of the Ricaati variable of wave function, and a new universality has been found in chaotic and disordered systems [20]. The amplitude distribution of an eigenfunction in a classically chaotic quantum billiard system is conjectured to be Gaussian in the semiclassical limit by Berry [2]. The numerical studies in very high-lying eigenstates support this conjecture [10,11,15]. In fact, in the far-semiclassical limit, even in scarred eigenstates, this conjecture remains true [15].

The correlation function of the wavefunction reveals some local fluctuation properties of the eigenstates. Following Berry's conjecture [2] that the wavefunction for a chaotic billiard is a superposition of infinite plane waves with random coefficients, the two-point correlation function is found to be proportional to $J_0(kd)$ (J_0 is the Bessel function of the zeroth order, k the wave vector and d the distance between the two points). The numerical calculations in high-lying chaotic eigenstates reveal some discrepancy with this conclusion both in a quantum system whose classical counterpart is the geodesic motion on a compact surface of constant negative curvature (as studied by Aurich and Steiner [10]) and in the Limaçon billiard (as studied by Li and Robnik [11]). Later on, Srednicki and Stiernelof [14] investigated this fluctuation from a statistical point of view.

In a detailed study of the autocorrelation function of an individual eigenstate, Li and Robnik [11] found that the two-point correlation function is direction dependent. Only after averaging over many directions, do the numerical calculations agree with Berry's prediction. To date, a satisfactory explanation for this discrepancy has been lacking. This motivated us to study analytically the two-point correlation function of eigenstates in a chaotic quantum billiard in terms of periodic orbit theory.

The billiard system we are considering is a quantum system of one particle moving in a closed domain \mathcal{B} with area \mathcal{A} and boundary $\partial\mathcal{B}$. The potential vanishes inside the billiard and has infinite value otherwise, $V(\mathbf{r}) = 0$ for all $\mathbf{r} \in \mathcal{B}$ and $V(\mathbf{r}) = \infty$ for all $\mathbf{r} \in \partial\mathcal{B}$. The quantum eigenstate of such a billiard satisfies the Schrödinger equation (Helmholtz equation),

$$\left(\frac{\hbar^2}{2m} \nabla^2 + E \right) \psi(\mathbf{r}) = 0, \quad (1)$$

with Dirichlet boundary condition, $\psi(\mathbf{r}) = 0$ for $\mathbf{r} \in \partial\mathcal{B}$, and is normalized, i.e., $\int_{\mathcal{A}} d\mathbf{r} |\psi(\mathbf{r})|^2 = 1$.

In order to define a two-point correlation function we first consider a point \mathbf{r} in some subset of \mathcal{B} , say \mathcal{C} , with area given by $\mathcal{A}_{\mathcal{C}}$. We will impose two conditions on \mathcal{C} . First, all points in \mathcal{C} are said to be far from the boundary, $\partial\mathcal{B}$. Second, we fix $\mathcal{A}_{\mathcal{C}}$ such that $\mathcal{A}_{\mathcal{C}} \ll \mathcal{A}$. Third, we consider a real vector, \mathbf{d} , such that for all $\mathbf{r} \in \mathcal{C}$, $\mathbf{r} \pm \mathbf{d}/2$ lies within the classically allowed region but not necessarily in \mathcal{C} . We then introduce a function given by

$$C(\mathbf{r}, \mathbf{d}; \Delta E) = \frac{\sum_n \psi_n(\mathbf{r} + \mathbf{d}/2) \psi_n^*(\mathbf{r} - \mathbf{d}/2)}{\sum_n |\psi_n(\mathbf{r})|^2}. \quad (2)$$

We also consider the position-smoothed object given by

$$C(\mathcal{C}, \mathbf{d}; \Delta E) = \frac{\int_{\mathcal{C}} d\mathbf{r} \sum_n \psi_n(\mathbf{r} + \mathbf{d}/2) \psi_n^*(\mathbf{r} - \mathbf{d}/2)}{\int_{\mathcal{C}} d\mathbf{r} \sum_n |\psi_n(\mathbf{r})|^2}. \quad (3)$$

Both definitions correspond to a two-point correlation function (the two points are $(\mathbf{r} = \pm \mathbf{d}/2)$) of a group of eigenstates. In the first case, it is a function of a point in \mathcal{C} , \mathbf{r} , an energy window, ΔE , and a real vector, \mathbf{d} ; in the second case we perform a position smoothing in \mathbf{r} over the region \mathcal{C} . Let N be the number of eigenstates whose eigenenergies lie in ΔE ; all N eigenstates must appear in the sum in equations (2) and (3). We also define an energy E_0 such that

$$E_0 - \frac{1}{2}\Delta E \leq E_n \leq E_0 + \frac{1}{2}\Delta E$$

for all n in the set of eigenstates.

Thus, from the outset we consider an energy-averaged (energy-smoothed) object as in [7] but may, in the end, consider the limit of small ΔE such that $N = 1$. We will also mention that the correlation function with energy smoothing has also been widely used in the study of disordered systems (for a review please see Ref. [23] and the references therein).

Historically the autocorrelation function in quantum chaos has revolved around the work of Berry. Berry's original definition [2] is similar to equation (3) but considers strictly one eigenstate and imposes a technical but important restriction on the area, $\mathcal{A}_{\mathcal{C}}$. Berry considers the function given by

$$C(\mathcal{C}, \mathbf{d}; E = E_n) = \frac{\int_{\mathcal{C}} d\mathbf{r} \psi_n(\mathbf{r} + \mathbf{d}/2) \psi_n^*(\mathbf{r} - \mathbf{d}/2)}{\int_{\mathcal{C}} d\mathbf{r} |\psi_n(\mathbf{r})|^2} \quad (4)$$

with the area, $\mathcal{A}_{\mathcal{C}}$ is mentioned in the following.

From the outset, the quantity in equation (4) is chosen to be a function of one eigenstate, E_n , rather than of an energy window containing one or more eigenstates. As for \mathcal{C} , it is chosen such that its volume tends to zero in the semiclassical limit [2], i.e. $\lim_{\hbar \rightarrow 0} \mathcal{A}_{\mathcal{C}} = 0$ but $\lim_{\hbar \rightarrow 0} \hbar / \sqrt{\mathcal{A}_{\mathcal{C}}} = 0$. As pointed out by Berry [5], the vanishing of $\mathcal{A}_{\mathcal{C}}$ ensures the smoothed functions to be 'semiclassically sharp', whereas $\sqrt{\mathcal{A}_{\mathcal{C}}}$ vanishes more slowly than \hbar so that the oscillatory detail on the scale of the de Broglie wavelength is smoothed away. This crucial point also allows one to reduce equation (4) to a Bessel function, $J_0(kd)$, where $k = \sqrt{2mE}/\hbar$ is the de Broglie wave number. Li and Robnik's numerical calculations are based on equation (4) and the condition given above.

The goal of this paper is to shed light on the above-mentioned direction-dependence problem of Berry's autocorrelation function by (4) working with the closely related definitions, equations (2) and (3), in the framework of periodic orbit theory. The introduction of an energy window, ΔE , will permit us to do what we want in a clear manner. Moreover, by using standard semiclassical methods, we will be able to calculate a first term given by $J_0(kd)$ and a second term which carries all directional dependencies of the vector \mathbf{d} .

The paper is organized as follows. In section 2 we give a general semiclassical formula for two-point correlation function. In section 3, the effect of energy averaging is discussed. The effect of position and direction averaging is discussed in sections. 4 and 5, respectively. The paper is concluded by a summary and discussion in section 6.

II. SEMICLASSICAL EXPRESSION OF TWO-POINT CORRELATION FUNCTION

To develop equations (2) or (3), let us begin by introducing the Green function in the energy representation corresponding to equation (1). It satisfies the following equation,

$$\left(E + \frac{\hbar^2}{2m} \nabla_{\mathbf{r}''}\right) G(\mathbf{r}'', \mathbf{r}'; E) = \delta(\mathbf{r}'' - \mathbf{r}'). \quad (5)$$

Using the series expansion for $G(\mathbf{r}'', \mathbf{r}'; E)$ in terms of the eigenstates of the Hamiltonian:

$$G(\mathbf{r}'', \mathbf{r}'; E) = \sum_n \frac{\psi_n^*(\mathbf{r}'') \psi_n(\mathbf{r}')}{E - E_n + i\epsilon} \quad (6)$$

where $\epsilon > 0$, it is easy to recast equation (2) in terms of the Green function,

$$\sum_n \psi_n(\mathbf{r}') \psi_n^*(\mathbf{r}'') \delta(E - E_n) = \frac{1}{2\pi i} [G^*(\mathbf{r}', \mathbf{r}'', E) - G(\mathbf{r}'', \mathbf{r}', E)] \quad (7)$$

Take the integral over energy integral $[E_0 - \Delta E/2, E_0 + \Delta E/2]$ on both sides we have:

$$\sum_n \psi_n(\mathbf{r}') \psi_n^*(\mathbf{r}'') = \frac{1}{2\pi i} [\langle G^*(\mathbf{r}', \mathbf{r}'', E) - G(\mathbf{r}'', \mathbf{r}', E) \rangle] \quad (8)$$

where $\langle f(E) \rangle$ represents an energy smoothing or an integral on the energy window, ΔE , containing the N energy levels,

$$\langle f(E) \rangle = \frac{1}{\Delta E} \int_{E_0 - \Delta E/2}^{E_0 + \Delta E/2} dE' f(E').$$

Thus we have:

$$\sum_n |\psi_n(\mathbf{r})|^2 = \frac{1}{2\pi i} [\langle G^*(\mathbf{r}, \mathbf{r}, E) - G(\mathbf{r}, \mathbf{r}, E) \rangle] = -\frac{1}{\pi} \Im m \langle G(\mathbf{r}, \mathbf{r}; E) \rangle, \quad (9)$$

$$C(\mathbf{r}, \mathbf{d}; \Delta E) = \frac{1}{2i} \frac{\langle G(\mathbf{r} + \mathbf{d}/2, \mathbf{r} - \mathbf{d}/2; E) \rangle - \langle G^*(\mathbf{r} - \mathbf{d}/2, \mathbf{r} + \mathbf{d}/2; E) \rangle}{\Im m \langle G(\mathbf{r}, \mathbf{r}; E) \rangle} \quad (10)$$

and the position smoothed object,

$$C(\mathcal{C}, \mathbf{d}; \Delta E) = \frac{1}{2i} \frac{\int_{\mathcal{C}} d\mathbf{r} \langle G(\mathbf{r} + \mathbf{d}/2, \mathbf{r} - \mathbf{d}/2; E) \rangle - \langle G^*(\mathbf{r} - \mathbf{d}/2, \mathbf{r} + \mathbf{d}/2; E) \rangle}{\int_{\mathcal{C}} d\mathbf{r} \Im m \langle G(\mathbf{r}, \mathbf{r}; E) \rangle} \quad (11)$$

In the semiclassical limit, the Green function is decomposed into a sum over two independent contributions:

$$G(\mathbf{r}'', \mathbf{r}'; E) = G_{DI}(\mathbf{r}'', \mathbf{r}'; E) + G_{IN}(\mathbf{r}'', \mathbf{r}'; E) + \mathcal{O}\left(\frac{1}{\hbar}\right). \quad (12)$$

The first term G_{DI} on the right represents a contribution from short-range trajectories passing directly from \mathbf{r}' to \mathbf{r}'' . The second term D_{IN} represents a long-range contribution (i.e. they make one or more bounces from the billiard wall) from classical trajectories beginning and finishing at these points.

Now we shall calculate the numerator and denominator in equations (10) and (11) separately. The denominator,

$$\Im m \langle G(\mathbf{r}, \mathbf{r}; E) \rangle = \Im m \langle G_{DI}(\mathbf{r}, \mathbf{r}; E) \rangle + \Im m \langle G_{IN}(\mathbf{r}, \mathbf{r}; E) \rangle. \quad (13)$$

Given,

$$\Im m G_{DI}(\mathbf{r}, \mathbf{r}; E) = -\pi \int \delta[E - H(\mathbf{p}, \mathbf{r})] \frac{d\mathbf{p}}{(2\pi\hbar)^2}$$

and that for a free particle moving inside the quantum billiard, $H(\mathbf{p}, \mathbf{r}) = \mathbf{p}^2/2m$, we have

$$\Im m \langle G_{DI}(\mathbf{r}, \mathbf{r}; E) \rangle = -\frac{m}{2\hbar^2} \quad (14)$$

The contribution from long-range trajectories (indirect trajectories) is given by the conventional sum over classical trajectories starting and ending at \mathbf{r}

$$G_{IN}(\mathbf{r}, \mathbf{r}; E) = \frac{m}{i\hbar(2\pi i\hbar)^{1/2}} \sum_{cl} |\Delta|^{1/2} \exp\left(\frac{i}{\hbar} S(\mathbf{r}, \mathbf{r}; E) - i\gamma \frac{\pi}{2}\right), \quad (15)$$

$$\Delta = \mathbf{det} \left(\frac{1}{|\mathbf{v}''||\mathbf{v}'|} \frac{\partial^2 S(\mathbf{r}'', \mathbf{r}'; E)}{\partial t'' \partial t'} \right)_{t'=t''=0; \mathbf{r}'=\mathbf{r}''=\mathbf{r}}.$$

Here γ is the count along the trajectory where the determinant (Δ_d) of second derivatives of the action in the pre-factor diverges. $t'(t'')$ measures a distance perpendicular to the trajectory at the initial (final) point and $v'(v'')$ is the velocity of the trajectory at the initial (final) point. The definition of the coordinates t will become evident shortly.

Equations (10) and (11) give the general formulae for two-point correlation function. In the subsequent sections we will evaluate these two formulae by different smoothing.

III. THE EFFECT OF ENERGY AVERAGING

For the rest of the discussion we will write a subscript, d , for all data referring to the trajectory, $\mathbf{r} \pm \mathbf{d}/2 \rightarrow \mathbf{r} \mp \mathbf{d}/2$, and 0 for all data referring to the closed trajectory, $\mathbf{r} \rightarrow \mathbf{r}$. Thus, $S_0 \equiv S_d(\mathbf{r}, \mathbf{r}; E)$ and $S_d \equiv S_d(\mathbf{r} \pm \mathbf{d}/2, \mathbf{r} \mp \mathbf{d}/2; E)$. We have

$$\Im m \langle G_{IN}(\mathbf{r}, \mathbf{r}; E) \rangle = -\frac{m}{\hbar\sqrt{2\pi\hbar}} \sum_{cl} |\Delta_0|^{1/2} \cos\left(\frac{S_0}{\hbar} - \frac{\pi}{4}(2\gamma + 1)\right) \quad (16)$$

Using equation (14), the denominator (13) becomes

$$\Im m \langle G(\mathbf{r}, \mathbf{r}; E) \rangle = -\frac{m}{2\hbar^2} \left[1 + \sqrt{\hbar} \frac{2}{\sqrt{2\pi}} \sum_{cl} |\Delta_0|^{1/2} \langle \cos\left(\frac{S_0}{\hbar} - \frac{\pi}{4}(2\gamma + 1)\right) \rangle \right] \quad (17)$$

By the same procedure we can also calculate the numerator, we denote it as I_N ,

$$\begin{aligned} I_N &= \langle G(\mathbf{r} + \mathbf{d}/2, \mathbf{r} - \mathbf{d}/2; E) - G^*(\mathbf{r} - \mathbf{d}/2, \mathbf{r} + \mathbf{d}/2; E) \rangle \\ &= I_N^{DI} + I_N^{IN} \end{aligned} \quad (18)$$

where

$$\begin{aligned} I_N^{DI} &= \langle G_{DI}(\mathbf{r} + \mathbf{d}/2, \mathbf{r} - \mathbf{d}/2; E) - G_{DI}^*(\mathbf{r} - \mathbf{d}/2, \mathbf{r} + \mathbf{d}/2; E) \rangle \\ I_N^{IN} &= \langle G_{IN}(\mathbf{r} + \mathbf{d}/2, \mathbf{r} - \mathbf{d}/2; E) - G_{IN}^*(\mathbf{r} - \mathbf{d}/2, \mathbf{r} + \mathbf{d}/2; E) \rangle \end{aligned} \quad (19)$$

For a quantum billiard the short-range Green function is given by the usual formula,

$$G_{DI}(\mathbf{r}'', \mathbf{r}'; E) = \frac{m}{2i\hbar^2} H_0(k|\mathbf{r}'' - \mathbf{r}'|) \quad (20)$$

where $H_0(x)$ is the zeroth order Hankel function of the first kind and $p = |\mathbf{p}| = \sqrt{2mE}$, $k = p/\hbar$. Posing $\mathbf{r}'' = \mathbf{r} + \mathbf{d}/2$ and $\mathbf{r}' = \mathbf{r} - \mathbf{d}/2$ we get

$$G_{DI}(\mathbf{r} \pm \mathbf{d}/2, \mathbf{r} \mp \mathbf{d}/2, E) = \frac{m}{2i\hbar^2} H_0(kd). \quad (21)$$

Note that this Green function depends only on the position coordinates via $d = |\mathbf{r}'' - \mathbf{r}'|$ and satisfies the natural boundary condition. The first term in equation (18) then becomes,

$$I_N^{DI} = -i \frac{m}{\hbar^2} \langle J_0(kd) \rangle \quad (22)$$

and the second term,

$$I_N^{IN} = \frac{2m}{i\hbar\sqrt{2\pi\hbar}} \sum_{cl} |\Delta_d|^{1/2} \langle \cos\left(\frac{S_d}{\hbar} - \frac{\pi}{4}(2\gamma + 1)\right) \rangle \quad (23)$$

Thus,

$$I_N = \frac{m}{i\hbar^2} \left[\langle J_0(kd) \rangle + \sqrt{\hbar} \frac{2}{\sqrt{2\pi}} \sum_{cl} |\Delta_d|^{1/2} \langle \cos\left(\frac{S_d}{\hbar} - \frac{\pi}{4}(2\gamma + 1)\right) \rangle \right] \quad (24)$$

We should point out that to obtain the above result, we have used the system's time-reversal symmetry whereby the trajectory, $\mathbf{r} + \mathbf{d}/2 \rightarrow \mathbf{r} - \mathbf{d}/2$, has the same classical properties entering equation (18) as the trajectory $\mathbf{r} - \mathbf{d}/2 \rightarrow \mathbf{r} + \mathbf{d}/2$.

Substituting the denominator, equation (17), and numerator, equation (24), together into equation (12) we have finally an expression for equation (2) up to the leading term of $\sqrt{\hbar}$,

$$\begin{aligned} C(\mathbf{r}, \mathbf{d}; \Delta E) &= \langle J_0(kd) \rangle + \sqrt{\hbar} \sqrt{\frac{2}{\pi}} \sum_{cl} |\Delta_d|^{1/2} \langle \cos\left(\frac{S_d}{\hbar} - \frac{\pi}{4}(2\gamma + 1)\right) \rangle \\ &\quad - \sqrt{\hbar} \sqrt{\frac{2}{\pi}} \sum_{cl} |\Delta_0|^{1/2} \langle \cos\left(\frac{S_0}{\hbar} - \frac{\pi}{4}(2\gamma + 1)\right) \rangle \end{aligned} \quad (25)$$

The sum in the second term in the above equation is over all trajectories connecting $\mathbf{r} + \mathbf{d}/2$ and $\mathbf{r} - \mathbf{d}/2$. In the third term it is over all closed trajectories starting and finishing at \mathbf{r} . The energy smoothing has the subsequent affect of giving preference to trajectories with a time of motion satisfying, $\tau < \hbar/\Delta E$. Thus as we reduce the value of ΔE we are forced to add an increasing number of trajectories in equation (25) to maintain numerical consistency and accuracy. Conversely, by choosing an average over many states we are free to choose less trajectories in the sum above, an evident advantage from the numerical point of view.

IV. THE EFFECT OF POSITION AVERAGING

Now we consider the case where numerator and denominator in equation (2) are to be integrated over some small region \mathcal{C} with area, $\mathcal{A}_{\mathcal{C}}$. This amounts to integrating equation (17) and equation (18) over \mathcal{C} . In order to proceed we will first consider the possibility that $\mathcal{A}_{\mathcal{C}}$ is either fixed (in \hbar) or goes to zero so slowly in the semiclassical limit that the long-range Green function has many rapid oscillations on \mathcal{C} .

With the above-mentioned simplification, the long-range Green function can be integrated via the method of stationary phase. The integral of interest has the general form,

$$\int dx \int dy \sum_{cl} e^{iS_{cl}(\mathbf{r} \pm \mathbf{d}/2, \mathbf{r} \mp \mathbf{d}/2; E)/\hbar}.$$

The stationary phase points in \mathcal{C} , labeled by $\bar{\mathbf{r}}$, are solutions of the equation,

$$\left(\frac{\partial S}{\partial \mathbf{r}''} \right)_{(\bar{\mathbf{r}} \pm \mathbf{d}/2, \bar{\mathbf{r}} \mp \mathbf{d}/2)} + \left(\frac{\partial S}{\partial \mathbf{r}'} \right)_{(\bar{\mathbf{r}} \pm \mathbf{d}/2, \bar{\mathbf{r}} \mp \mathbf{d}/2)} = 0. \quad (26)$$

The solutions of equation (26) correspond to points $\bar{\mathbf{r}}$ in \mathcal{C} for which there exists a trajectory leaving a point $\mathbf{r}' = \bar{\mathbf{r}} \pm \mathbf{d}/2$ and arriving at a point $\mathbf{r}'' = \bar{\mathbf{r}} \mp \mathbf{d}/2$ with equal initial and final momenta in x and y . This translates into a search for points $\bar{\mathbf{r}}$ in \mathcal{C} affiliated with a trajectory which has the following two properties:

- (a) its path includes two parallel chords
- (b) on one chord there exists a point, $\bar{\mathbf{r}} \pm \mathbf{d}/2$, and on the other a point, $\bar{\mathbf{r}} \mp \mathbf{d}/2$.

The first condition assures that the initial and final momenta are equal. The second condition assures that a vector, \mathbf{d} , can be oriented between the two chords with midpoint given by $\bar{\mathbf{r}}$. Geometrically, we search for a trajectory whose path includes two parallel chords that lie on opposite sides (the perpendicular distance is given by $\mathbf{d}_{\perp}/2$) of a third chord also running parallel to the two outer chords and lying in \mathcal{C} (please see Fig.1). It is not important that the two chords lie in \mathcal{C} , however, by definition $\bar{\mathbf{r}}$ must lie on the midpoint chord which lies in this region. It is also clear that the solutions of equation (26) are not isolated; in fact they lie on a line, the midpoint chord. We speak of a line of stationary phase points, S , and to treat the integration in x and y we introduce the usual system of coordinates to describe motion local to a trajectory. Moreover, due to time-invariance a solution corresponding to the path $\bar{\mathbf{r}} + \mathbf{d}/2 \rightarrow \bar{\mathbf{r}} - \mathbf{d}/2$ will also be a solution corresponding to the path $\bar{\mathbf{r}} - \mathbf{d}/2 \rightarrow \bar{\mathbf{r}} + \mathbf{d}/2$.

There exist two types of trajectories that satisfy the required conditions (see Fig. (1)). Trivially any periodic orbit which has a chord in \mathcal{C} parallel to \mathbf{d} will satisfy this condition and its stationary phase points then lie on that chord (Fig. 1a). A second possibility comes from non-periodic trajectories (Fig. 1b). In both cases $v'' = v' = \dot{s}$. Having found the stationary phase points, one can expand the action to second order in both x and y . On the other hand, for the purposes of integrating over \mathcal{C} it is most convenient to introduce the usual Cartesian set of local coordinates, describing motion parallel and perpendicular to S . Let us denote this new set as (s, t) with $|\partial(x, y)/\partial(s, t)| = 1$. The variable t measures a displacement perpendicular to S and s measures a displacement along S . For case (1) trajectories the point $\bar{\mathbf{r}}$ lies on the trajectory itself but in the second case it lies between two chords that are parallel to one another (and to S) and are separated by $\mathbf{d}_\perp/2$.

For both cases (1) and (2), one now proceeds to integrate over s and t by expanding the action to second order in t (in both initial and final point), for fixed s , and integrating first in t . Expansion of the action to second order in t gives,

$$\begin{aligned} S((s + d_\parallel/2, t + d_\perp/2), (s - d_\parallel/2, t - d_\perp/2); E) \\ = S((s + d_\parallel/2, d_\perp/2), (s - d_\parallel/2, -d_\perp/2); E) \\ + \frac{1}{2} \left(\frac{\partial^2 S}{\partial t'^2} + \frac{\partial^2 S}{\partial t''^2} + 2 \frac{\partial^2 S}{\partial t' \partial t''} \right) t^2 + O(t^3). \end{aligned} \quad (27)$$

The first term refers to the action, S_d , for going from the point $(s + d_\parallel/2, d_\perp/2)$ to the point $(s - d_\parallel/2, -d_\perp/2)$. It is clearly invariant in s , the coordinate along S . Fixing s at one value on S , we can define the linear Poincaré map, $dP((t', p') \rightarrow (t'', p''))$. It is a matrix with four elements given by:

$$\begin{aligned} \delta t'' &= m_{11} \delta t' + m_{12} \delta p' \\ \delta p'' &= m_{21} \delta t' + m_{22} \delta p'. \end{aligned} \quad (28)$$

We also have the well known relations [24]

$$\frac{\partial^2 S}{\partial t'^2} = \frac{m_{11}}{m_{12}}, \quad \frac{\partial^2 S}{\partial t''^2} = \frac{m_{22}}{m_{12}}, \quad \frac{\partial^2 S}{\partial t' \partial t''} = -\frac{1}{m_{12}} \quad (29)$$

which we use to write the second variation as $(m_{11} + m_{22} - 2)/(2m_{12})$. Again in equation (28) we are considering variations in t at the initial and final points.

The indirect part of the Green function is given by [24],

$$G_{IN}(\mathbf{r}'', \mathbf{r}'; E) = \frac{m}{i\hbar(2\pi i\hbar)^{1/2}} \sum_d \frac{|D(s)|^{1/2}}{|v|} \exp \left(\frac{i}{\hbar} \left(S_d + \frac{W(s)}{2} t^2 \right) - i\gamma \frac{\pi}{2} \right), \quad (30)$$

where

$$D(s) = \frac{1}{m_{12}^d}, \quad W(s) = \frac{m_{11}^d + m_{22}^d - 2}{m_{12}^d}. \quad (31)$$

Again the symbol d refers to classical motion beginning at $\bar{\mathbf{r}} \pm \mathbf{d}/2$ and finishing at $\bar{\mathbf{r}} \mp \mathbf{d}/2$ whereas the symbol 0 refers to the special case, $\mathbf{d} = 0$.

The integration over t gives rise to

$$\int_{-\infty}^{+\infty} dt \exp\left(\frac{iW(s)}{2\hbar}t^2\right) = \sqrt{\frac{2\pi\hbar}{|m_{11}^d + m_{22}^d - 2|}} \exp\left(i\frac{\pi}{4}\text{sign}(m_{11}^d + m_{22}^d - 2)\right) \quad (32)$$

Here the sign function is given by;

$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \quad (33)$$

Due to the invariance of action and the term $m_{11} + m_{22}$ the subsequent integration in s is particularly simple; from it we get the length of S in \mathcal{C} divided by the speed along it, $|v|$, in other words a time which we denote T_{cl}^d . This gives

$$\int_{\mathcal{A}_c} d\mathbf{r} \langle G_{IN}(\mathbf{r} \pm \mathbf{d}/2, \mathbf{r} \mp \mathbf{d}/2, E) \rangle = -i\sqrt{\hbar}m \sum_{cl} \frac{T_{cl}}{\sqrt{|m_{11}^d + m_{22}^d - 2|}} \langle \exp\left(\frac{i}{\hbar}S_d - \frac{i}{4}\pi\sigma_d\right) \rangle \quad (34)$$

where

$$\sigma_d = 2\gamma_d + 1 - \sigma(m_{11}^d + m_{22}^d - 2)$$

Therefore,

$$\begin{aligned} \int_{\mathcal{A}_c} d\mathbf{r} \langle G_{IN}(\mathbf{r} + \mathbf{d}/2, \mathbf{r} - \mathbf{d}/2, E) - G_{IN}^*(\mathbf{r} - \mathbf{d}/2, \mathbf{r} + \mathbf{d}/2, E) \rangle \\ = -i\sqrt{\hbar}m \sum_{cl} \frac{T_{cl}^d}{\sqrt{|m_{11}^d + m_{22}^d - 2|}} \langle \cos\left(\frac{S_d}{\hbar} - \frac{\sigma_d}{4}\pi\right) \rangle \end{aligned} \quad (35)$$

The numerator is thus,

$$I_N = \frac{m\mathcal{A}_c}{i\hbar^2} \left[\langle J_0(kd) \rangle + \frac{2\hbar}{\mathcal{A}_c} \sum_{cl} \frac{T_{cl}^d}{\sqrt{|m_{11}^d + m_{22}^d - 2|}} \langle \cos\left(\frac{S_d}{\hbar} - \frac{\sigma_d}{4}\pi\right) \rangle \right] \quad (36)$$

Similarly, we can also obtain the expression for the denominator

$$\int_{\mathcal{C}} d\mathbf{r} \Im m \langle G(\mathbf{r}, \mathbf{r}; E) \rangle = -\frac{m\mathcal{A}_c}{2\hbar^2} \left[1 + \frac{\hbar}{\mathcal{A}_c} \sum_{cl} \frac{T_{cl}^0}{\sqrt{|m_{11}^0 + m_{22}^0 - 2|}} \langle \cos\left(\frac{S_0}{\hbar} - \frac{\sigma_0}{4}\pi\right) \rangle \right] \quad (37)$$

Finally the correlation function is given by,

$$\begin{aligned} C(\mathcal{C}, \mathbf{d}; \Delta E) = \langle J_0(kd) \rangle + \hbar \frac{2}{\mathcal{A}_c} \sum_{cl} \frac{T_{cl}^d}{\sqrt{|m_{11}^d + m_{22}^d - 2|}} \langle \cos\left(\frac{S_d}{\hbar} - \frac{\sigma_d}{4}\pi\right) \rangle \\ - \hbar \frac{2}{\mathcal{A}_c} \sum_{cl} \frac{T_{cl}^0}{\sqrt{|m_{11}^0 + m_{22}^0 - 2|}} \langle \cos\left(\frac{S_0}{\hbar} - \frac{\sigma_0}{4}\pi\right) \rangle \end{aligned} \quad (38)$$

The sum in the second term is over all trajectories satisfying the criteria (a) and (b). *It carries the directional dependency of \mathbf{d} via the choice of trajectories.* The sum in the third term is over all periodic orbits starting and ending at the stationary point in the averaging area of \mathcal{A}_C , this term does not depend on the direction. All trajectories must be included, however, the subsequent energy smoothing will restrict this sum to trajectories whose time scale is $\tau \approx \hbar/\Delta E$. Again our final answer cannot depend on ΔE but this factor will control the contribution of each trajectory in the sum. Trajectories whose time-scales greatly exceed τ will be washed out by the energy smoothing. Any periodic orbit which has a chord parallel to \mathbf{d} in \mathcal{C} will contribute to this term. However, a periodic orbit which scars a particular state whose energy is outside the window, ΔE , may not contribute.

V. THE EFFECT OF DIRECTION AVERAGING

In this section, we will consider the sum in equation (38) over different directions of the vector \mathbf{d} . To do this one can picture a quantum billiard with vector \mathbf{d} centered at $(x, y) = (0, 0)$. Rotation of \mathbf{d} produces a circle of radius $d/2$ centered at $(0, 0)$. Each trajectory in equations (25) and/or (38) begins and finishes on this circle. The initial and final point are separated by the diameter d . Most generally, the dynamics defines a map P which is a function of initial point $\mathbf{q} = (x, y)$, momentum $\mathbf{p} = (p_x, p_y)$ and time. The linear map (DP) is a 4*4 matrix.

To enumerate trajectories one constructs a symbolic dynamics [25]. This construction allows one to assign to each trajectory in equations (25) and/or (38), a sequence of numbers (e.g. a code (i)) and a smooth function S_i , the action of that trajectory. Given a trajectory labeled by i which starts at θ^* and finishes at $\theta^* + \pi$, the idea now is to find a new trajectory which runs almost the same pattern but begins at $\theta = \theta^* + d\theta$ and finishes at $\theta' = \theta + \pi$. This matching can be performed by using the linear map, DP (with matrix elements given by the original trajectory), substituting $\delta x = -d \sin \theta \delta \theta$, and $\delta y = d \cos \theta \delta \theta$, and solving for initial $(\delta p_x, \delta p_y)$. This method for matching can be continued until the linear map has a singular point. Clearly S_i , the action built for code i , is a function of θ for all i .

Let us now denote terms (1),(2) and (3) in equations (25) and/or (38) by f_o, f_1 and f_2 , respectively. We also define an average over θ given by

$$\tilde{f}(\theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} f d\theta.$$

Clearly f_o and f_2 are independent of θ and can be trivially integrated; they remain of order 1 and $\sqrt{\hbar}$ respectively. On the other hand, on averaging f_1 in the semiclassical limit, only points θ^* where $\partial S_i / \partial \theta = 0$ will give useful results. After Gaussian integration, any term labeled by (i) in \tilde{f}_1 , will either be of order \hbar if θ^* exists in the domain of integration or zero otherwise. Thus in the semiclassical limit a sum over many θ would semiclassically clear away corrections of order $\sqrt{\hbar}$ which are θ dependent (e.g. the term f_1). Moreover, note that θ_1 and θ_2 need not satisfy $\theta_2 - \theta_1 = 2\pi$.

In the paper [11], numerical work on a single high-lying eigenfunction in the Limaçon billiard was performed. It was shown there that while Berry's conjecture was not satisfied for a fixed θ , a direction average over many θ (with $0 \leq \theta < 2\pi$) appeared to give a result

which exactly corresponded to that of Berry's conjecture. This latter point would intuitively suggest that \tilde{f}_1 exactly cancels \tilde{f}_2 using $\theta_1 = 0$ and $\theta_2 = 2\pi$,

$$\frac{1}{2\pi} \int_0^{2\pi} f_1(\theta) d\theta = -\tilde{f}_2 = -f_2,$$

leaving $f_o = \langle J_0(kd) \rangle$ as the only surviving contribution. But according to our arguments, \tilde{f}_1 is of order \hbar while \tilde{f}_2 is of order $\sqrt{\hbar}$. Hence these two terms cannot cancel each other. However, since f_1 is washed away by the direction average, the differences from $\tilde{f}_o = f_o = \langle J_o \rangle$ will come from f_2 alone.

VI. CONCLUSIONS

We have calculated a semiclassical expression for an (energy) averaged autocorrelation function over a group of eigenstates within the energy window ΔE up to the leading term of $\sqrt{\hbar}$. The final expression equations (25) and (38) are our main results for correlation functions equations (2) and (3), respectively. In both cases, the autocorrelation functions are expressed as a sum of three terms, the first one is the Bessel function of the zeroth order (Berry's term), the second and the third ones are of special trajectory modulations. In the first case, i.e. definition equation (2), the second term (in equation (25)) includes the contribution from all trajectories connecting the two ends of the vector $\mathbf{d} = \mathbf{r}'' - \mathbf{r}'$. The third term comes from the contribution of all *closed* trajectories starting and ending at \mathbf{r} . However, in the second case, i.e. have space averaging (equation (38)), things become quite different. The second term on the right-hand side of equation (38) is the contribution from all trajectories satisfying the criteria (a) and (b). This term carries the directional dependency of \mathbf{d} . There are two kinds of trajectories satisfy these two criteria: (1) any periodic orbit which has a chord in the averaging domain \mathcal{C} parallel to \mathbf{d} (see figure 1(a)) makes contribution; (2) any open trajectory (see figure 1(b)) having two segments parallel to each other and passing through (not necessarily perpendicular to the vector \mathbf{d}) the two ends of \mathbf{d} in the same direction contribute to this term. The averaging over all directions might wash away this term as observed numerically by Li and Robnik [11]. The third term on the right-hand side of equation (38) includes all periodic orbits starting and ending at the stationary point in the averaging domain \mathcal{C} . In principle, the number of trajectories to be calculated is infinite, however, the energy smoothing introduced from beginning restricts this to trajectories and/or periodic orbits whose time scale is about $\tau \approx \hbar/\Delta E$. This gives an advantage in numerical calculation.

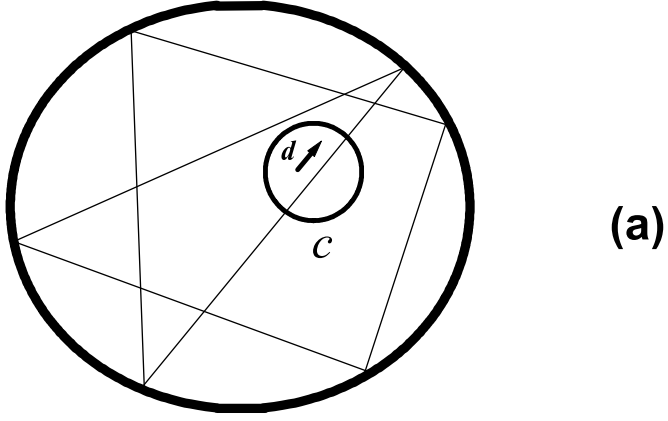
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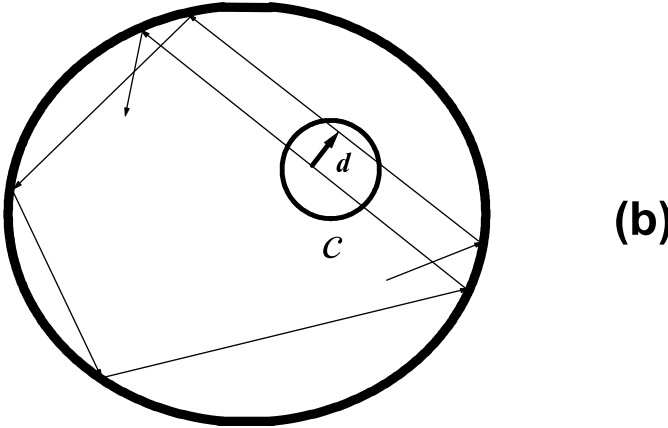
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FIGURES



(a)



(b)

FIG. 1. Schematic illustration of two different kinds of trajectories contributing to the second term on the right-hand side of equation (38). (a) Periodic orbit which has a chord in C parallel to $\mathbf{d} = \mathbf{r}'' - \mathbf{r}'$. (b) Non periodic orbit which has two segments parallel to each other and passing through the two ends of \mathbf{d} . It is obvious that the contribution depends on the direction of \mathbf{d} .